

# Partitioning 3-homogeneous latin bitrades

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## Abstract

A latin bitrade  $(T^\diamond, T^\otimes)$  is a pair of partial latin squares that define the difference between two arbitrary latin squares  $L^\diamond \supseteq T^\diamond$  and  $L^\otimes \supseteq T^\otimes$  of the same order. A 3-homogeneous bitrade  $(T^\diamond, T^\otimes)$  has three entries in each row, three entries in each column, and each symbol appears three times in  $T^\diamond$ . Cavenagh [2] showed that any 3-homogeneous bitrade may be partitioned into three transversals. In this paper we provide an independent proof of Cavenagh's result using geometric methods. In doing so we provide a framework for studying bitrades as tessellations in spherical, euclidean or hyperbolic space. Additionally, we show how latin bitrades are related to finite representations of certain triangle groups.

## 1 Introduction

A *latin bitrade*  $(T^\diamond, T^\otimes)$  is a pair of partial latin squares which are disjoint, occupy the same set of non-empty cells, and whose corresponding rows and columns contain the same set of entries. One of the earliest studies of latin bitrades appeared in [6], where they are referred to as *exchangeable partial groupoids*. Latin bitrades are prominent in the study of *critical sets*, which are minimal defining sets of latin squares ([1],[3],[9]) and the intersections between latin squares ([7]). We write  $i \diamond j = k$  when symbol  $k$  appears in the cell at the intersection of row  $i$  and column  $j$  of the (partial) latin square  $T^\diamond$ . A 3-homogeneous bitrade has 3 elements in each row, 3 elements in each column, and each symbol appears 3 times. Cavenagh [2] obtained the following theorem, using combinatorial methods, as a corollary to a general classification result on 3-homogeneous bitrades.

**Theorem 1.1** (Cavenagh [2]). *Let  $(T^\diamond, T^\otimes)$  be a 3-homogeneous bitrade. Then  $T^\diamond$  can be partitioned into three transversals.*

In this paper we provide an independent and geometric proof of Cavenagh's result. In doing so we provide a framework for studying bitrades as tessellations in spherical, euclidean or hyperbolic space. In particular, bitrades can be thought of as finite representations of certain triangle groups.

We let permutations act on the right, in accordance with computer algebra systems such as Sage [11]. Graphs in this paper may contain loops or multiple edges; otherwise our notation is standard and we refer the reader to Diestel [4]. Some basic topological terms will be used; for these we refer the reader to Stillwell [12]. Finally, a good reference for hypermaps and graphs on surfaces is [10].

## 2 Latin bitrades

A *partial latin square*  $P$  of order  $n > 0$  is an  $n \times n$  array where each  $e \in \{0, 1, \dots, n-1\}$  appears at most once in each row, and at most once in each column. A *latin square*  $L$  of order  $n > 0$  is an  $n \times n$  array where each  $e \in \{0, 1, \dots, n-1\}$  appears exactly once in each row, and exactly once in each column. It is convenient to use setwise notation to refer to entries of a (partial) latin square, and we write  $(i, j, k) \in P$  if and only if symbol  $k$  appears in the intersection of row  $i$  and column  $j$  of  $P$ . In this manner,  $P \subseteq A_1 \times A_2 \times A_3$  for finite sets  $A_i$ , each of size  $n$ . It is also convenient to interpret a (partial) latin square as a multiplication table for a binary operator  $\diamond$ , writing  $i \diamond j = k$  if and only if  $(i, j, k) \in T = T^\diamond$ .

**Definition 2.1.** Let  $T^\diamond, T^\otimes \subseteq A_1 \times A_2 \times A_3$  be two partial latin squares. Then  $(T^\diamond, T^\otimes)$  is a *bitrade* if the following three conditions are satisfied:

- (R1)  $T^\diamond \cap T^\otimes = \emptyset$ ;
- (R2) for all  $(a_1, a_2, a_3) \in T^\diamond$  and all  $r, s \in \{1, 2, 3\}$ ,  $r \neq s$ , there exists a unique  $(b_1, b_2, b_3) \in T^\otimes$  such that  $a_r = b_r$  and  $a_s = b_s$ ;
- (R3) for all  $(a_1, a_2, a_3) \in T^\otimes$  and all  $r, s \in \{1, 2, 3\}$ ,  $r \neq s$ , there exists a unique  $(b_1, b_2, b_3) \in T^\diamond$  such that  $a_r = b_r$  and  $a_s = b_s$ .

Conditions (R2) and (R3) imply that each row (column) of  $T^\diamond$  contains the same subset of  $A_3$  as the corresponding row (column) of  $T^\otimes$ . A *k-homogeneous bitrade*  $(T^\diamond, T^\otimes)$  has  $k$  entries in each row of  $T^\diamond$ ,  $k$  entries in each column of  $T^\diamond$ , and each symbol appears  $k$  times in  $T^\diamond$ . By symmetry the same holds for  $T^\otimes$ . A set  $\mathfrak{T} \subseteq T^\diamond$  is a *transversal* if  $\mathfrak{T}$  intersects each row of  $T^\diamond$  in precisely one entry, each column in precisely one entry, and if the number of symbols appearing in  $\mathfrak{T}$  is equal to  $|\mathfrak{T}|$ . The latter condition can be written as  $|\{k \mid (i, j, k) \in \mathfrak{T}\}| = |\mathfrak{T}|$ . A bitrade  $(T^\diamond, T^\otimes)$  is *primary* if whenever  $(U^\diamond, U^\otimes)$  is a bitrade such that  $U^\diamond \subseteq T^\diamond$  and  $U^\otimes \subseteq T^\otimes$ , then  $(T^\diamond, T^\otimes) = (U^\diamond, U^\otimes)$ . Bijections  $A_i \rightarrow A'_i$ , for  $i = 1, 2, 3$ , give an *isotopic* bitrade, and permuting each  $A_i$  gives an *autotopism*.

In [5], Drápal gave a representation of bitrades in terms of three permutations  $\tau_i$  acting on a finite set. For  $r \in \{1, 2, 3\}$ , define the map  $\beta_r: T^\otimes \rightarrow T^\diamond$  where  $(a_1, a_2, a_3)\beta_r = (b_1, b_2, b_3)$  if and only if  $a_r \neq b_r$  and  $a_i = b_i$  for  $i \neq r$ . By Definition 2.1 each  $\beta_r$  is a bijection. Then  $\tau_1, \tau_2, \tau_3: T^\diamond \rightarrow T^\diamond$  are defined by

$$\tau_1 = \beta_2^{-1}\beta_3, \quad \tau_2 = \beta_3^{-1}\beta_1, \quad \tau_3 = \beta_1^{-1}\beta_2. \quad (1)$$

We refer to  $[\tau_1, \tau_2, \tau_3]$  as the  $\tau_i$  representation. We write  $\text{Mov}(\pi)$  for the set of points that the (finite) permutation  $\pi$  acts on.

**Definition 2.2.** Let  $\tau_1, \tau_2, \tau_3$  be (finite) permutations and let  $\Omega = \text{Mov}(\tau_1) \cup \text{Mov}(\tau_2) \cup \text{Mov}(\tau_3)$ . Define four properties:

- (T1)  $\tau_1\tau_2\tau_3 = 1$ ;
- (T2) if  $\rho_i$  is a cycle of  $\tau_i$  and  $\rho_j$  is a cycle of  $\tau_j$  then  $|\text{Mov}(\rho_i) \cap \text{Mov}(\rho_j)| \leq 1$ , for any  $1 \leq i < j \leq 3$ ;
- (T3) each  $\tau_i$  is fixed-point-free;
- (T4) the group  $\langle \tau_1, \tau_2, \tau_3 \rangle$  is transitive on  $\Omega$ .

By letting  $A_i$  be the set of cycles of  $\tau_i$ , Drápal obtained the following theorem, which relates Definition 2.1 and 2.2.

**Theorem 2.3** (Drápal [5]). *A bitrade  $(T^\diamond, T^\otimes)$  is equivalent (up to isotopism) to three permutations  $\tau_1, \tau_2, \tau_3$  acting on a set  $\Omega$  satisfying (T1), (T2), and (T3). If (T4) is also satisfied then the bitrade is primary.*

To construct the  $\tau_i$  representation for a bitrade we simply evaluate Equation (1). In the reverse direction we have the following construction:

**Construction 2.4** ( $\tau_i$  to bitrade). Let  $\tau_1, \tau_2, \tau_3$  be permutations satisfying Condition (T1), (T2), and (T3). Let  $\Omega = \text{Mov}(\tau_1) \cup \text{Mov}(\tau_2) \cup \text{Mov}(\tau_3)$ . Define  $A_i = \{\rho \mid \rho \text{ is a cycle of } \tau_i\}$  for  $i = 1, 2, 3$ . Now define two arrays  $T^\diamond, T^\otimes$ :

$$\begin{aligned} T^\diamond &= \{(\rho_1, \rho_2, \rho_3) \mid \rho_i \in A_i \text{ and } |\text{Mov}(\rho_1) \cap \text{Mov}(\rho_2) \cap \text{Mov}(\rho_3)| \geq 1\} \\ T^\otimes &= \{(\rho_1, \rho_2, \rho_3) \mid \rho_i \in A_i \text{ and } x, x', x'' \text{ are distinct points of } \Omega \text{ such} \\ &\quad \text{that } x\rho_1 = x', x'\rho_2 = x'', x''\rho_3 = x\}. \end{aligned}$$

By Theorem 2.3  $(T^\diamond, T^\otimes)$  is a bitrade.

**Example 2.5.** The smallest bitrade  $(T^\diamond, T^\otimes)$  is the *intercalate*, which has four entries. The bitrade is shown below:

$$T^\diamond = \begin{array}{c|cc} \diamond & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \quad T^\otimes = \begin{array}{c|cc} \otimes & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}$$

The  $\tau_i$  representation is  $\tau_1 = (000, 011)(101, 110)$ ,  $\tau_2 = (000, 101)(011, 110)$ ,  $\tau_3 = (000, 110)(011, 101)$ , where we have written  $ijk$  for  $(i, j, k) \in T^\diamond$  to make

the presentation of the  $\tau_i$  permutations clearer. By Construction 2.4 with  $\Omega = \{000, 011, 101, 110\}$  we can convert the  $\tau_i$  representation to a bitrade  $(U^\diamond, U^\otimes)$ :

$$U^\diamond = \begin{array}{c|cc} \diamond & (000, 101) & (011, 110) \\ \hline (000, 011) & (000, 110) & (011, 101) \\ (101, 110) & (011, 101) & (000, 110) \end{array}$$

$$U^\otimes = \begin{array}{c|cc} \otimes & (000, 101) & (011, 110) \\ \hline (000, 011) & (011, 101) & (000, 110) \\ (101, 110) & (000, 110) & (011, 101) \end{array}$$

In this way we see that row 0 of  $T^\diamond$  corresponds to row (000, 011) of  $U^\diamond$ , which is the cycle (000, 011) of  $\tau_1$ , and so on for the columns and symbols.

**Example 2.6.** The following 3-homogeneous bitrade is pertinent to the proof of the main result of this paper:

$$T^\diamond = \begin{array}{c|cccc} \diamond & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 3 & & 2 \\ 2 & 3 & 2 & 4 & \\ 3 & & 4 & 3 & 1 \\ 4 & 2 & & 1 & 4 \end{array} \quad T^\otimes = \begin{array}{c|cccc} \otimes & 1 & 2 & 3 & 4 \\ \hline 1 & 3 & 2 & & 1 \\ 2 & 2 & 4 & 3 & \\ 3 & & 3 & 1 & 4 \\ 4 & 1 & & 4 & 2 \end{array} \quad (2)$$

Writing  $ijk \in T^\diamond$  for  $(i, j, k) \in T^\diamond$ , the  $\tau_i$  representation is

$$\begin{aligned} \tau_1 &= (111, 142, 123)(213, 234, 222)(324, 341, 333)(412, 444, 431) \\ \tau_2 &= (111, 213, 412)(123, 222, 324)(234, 333, 431)(142, 341, 444) \\ \tau_3 &= (111, 431, 341)(123, 333, 213)(142, 412, 222)(234, 444, 324) \\ \Omega &= \text{Mov}(\tau_1) \cup \text{Mov}(\tau_2) \cup \text{Mov}(\tau_3). \end{aligned}$$

The bitrade has four rows so  $\tau_1$  has four cycles; similarly  $\tau_2$  and  $\tau_3$  each have four cycles. (In general, a bitrade can have a different number of row, column, and symbol cycles.) Using Construction 2.4, the cell at row (111, 142, 123), column (111, 213, 412), will contain the symbol (111, 431, 341) since these cycles intersect in 111.

### 3 Bitrades as graphs on surfaces

Before showing how a bitrade can be represented as a graph embedded in a surface, we briefly review the theory of hypermaps. A *combinatorial hypermap*  $[\sigma, \alpha, \varphi]$  is made up of three permutations  $\sigma, \alpha, \varphi$  and a finite set  $\Omega$  such that  $\sigma\alpha\varphi = 1$  and  $G = \langle \sigma, \alpha \rangle$  acts transitively on  $\Omega$ . The following construction takes a combinatorial hypermap to a *hypermap*, which is a bipartite graph embedded in a surface. For a proof of correctness see Chapter 1 of [10] and references therein, and for further examples see Chapter 1 and 2 of [8]. The representation of hypermaps as bipartite graphs was given by Walsh [13].

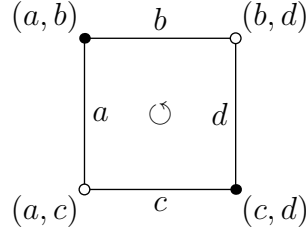


Figure 1: An embedding of a bipartite graph.

**Construction 3.1.** Let  $[\sigma, \alpha, \varphi]$  be a combinatorial hypermap on the finite set  $\Omega$ . Create vertex sets  $V_1, V_2$  and undirected edges  $E$ :

$$\begin{aligned} V_1 &= \{v \mid v \text{ is an cycle of } \sigma\} \\ V_2 &= \{v \mid v \text{ is an cycle of } \alpha\} \\ E &= \{\{v, v'\}_x \mid v \in V_1, v' \in V_2, \text{ and } x \in \text{Mov}(v) \cap \text{Mov}(v')\} \end{aligned}$$

Colour the vertices of  $V_1$  black (denoted  $\bullet$ ) and those of  $V_2$  white (denoted  $\circ$ ). When drawing the graph we usually label an edge  $\{v, v'\}_x$  with  $x$  to save space. Suppose that  $(x_1, x_2, \dots, x_n)$  is a cycle of  $\sigma$ , and let  $v$  be the associated black vertex with adjacent edges  $\{v, v_i\}_{x_i}$  for  $1 \leq i \leq n$ . Then order the edges adjacent to  $v$  as  $\{v, v_1\}_{x_1}, \{v, v_2\}_{x_2}, \dots, \{v, v_n\}_{x_n}$  in the anticlockwise direction. Apply the same process to each  $v \in V_2$ . This defines a *rotation scheme* for the vertices of the bipartite graph, and hence an embedding in a surface.

**Example 3.2.** Let  $\Omega = \{a, b, c, d\}$  and define  $\sigma = (a, b)(c, d)$  and  $\alpha = (a, c)(b, d)$ . Then there are two black vertices, two white vertices, and four edges:  $V_1 = \{(a, b), (c, d)\}$   $V_2 = \{(a, c), (b, d)\}$  and  $E = \{a, b, c, d\}$ . The graph embedding, with anticlockwise orientation, is shown in Figure 1.

Given a bipartite graph embedding, we often move to the *canonical triangulation*, as described in the following construction:

**Construction 3.3** ([10, p. 50]). Let  $\mathcal{H}$  be a hypermap. Place a new vertex  $\star$  in each face of the hypermap. Connect this new vertex to each vertex that lies on the border of the face using dotted edges to  $\bullet$  vertices and dashed edges to  $\circ$  vertices. The surface is now subdivided into triangles. Each triangle has three types of vertices:  $\bullet$ ,  $\circ$ , and  $\star$ ; each triangle has three types of sides: a solid, dashed, or dotted line. From the inside of a triangle, we view its vertices according to the order  $\bullet, \circ, \star, \bullet$ , and if we turn in the anticlockwise direction then the triangle is *positive*, otherwise it is *negative*. We shade the positive triangles.

Since each (shaded) triangle is adjacent to precisely one solid edge in the canonical triangulation, we can identify the action of  $\sigma$  as the rotation of shaded triangles around their black vertex in an anticlockwise direction, as shown in

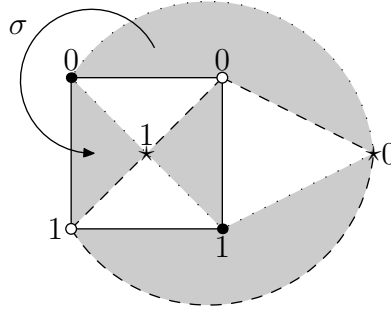


Figure 2: Canonical triangulation of the bipartite graph embedding of Figure 1. Vertices are labelled from the set  $\{0, 1\}$  to aid in identification with the intercalate bitrade, and the action of  $\sigma$  is shown on a particular shaded triangle.

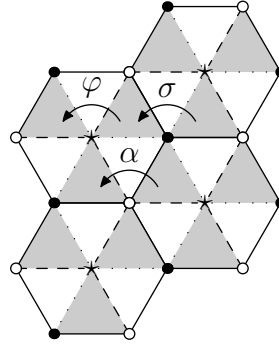


Figure 3: Canonical triangulation of a particular hypermap. The action of  $\sigma$ ,  $\alpha$ , and  $\varphi$  on certain shaded triangles is shown.

Figure 2 (also, see [10, p. 51]). In general, the action of  $\alpha$  and  $\varphi$  correspond to rotations around white and star vertices, as indicated in Figure 3.

**Example 3.4.** The canonical triangulation of the bipartite graph embedding of Example 3.2 is shown in Figure 2. Writing  $ijk$  for the shaded triangle with vertex labels  $i, j, k$  on black, white and star vertices, respectively, we see that  $000\sigma = 011$ ,  $000\alpha = 101$ , and  $000\varphi = 110$ . As expected, the action of  $\sigma$ ,  $\alpha$ , and  $\varphi$  in Figure 2 is exactly the same as  $\tau_1$ ,  $\tau_2$ , and  $\tau_3$  of Example 2.5.

Applying Euler's formula leads to the *genus formula* for hypermaps:

$$z(\sigma) + z(\alpha) + z(\varphi) - |\Omega| = 2 - 2g \quad (3)$$

where  $z(\pi)$  denotes the number of cycle of the permutation  $\pi$ .

**Lemma 3.5.** *A 3-homogeneous bitrade  $(T^\diamond, T^\otimes)$  defines a tessellation of shaded and unshaded triangles in the Euclidean plane. Each shaded triangle is edge-wise adjacent only to unshaded triangles (and vice-versa). Shaded and unshaded triangles correspond to the entries of  $T^\diamond$  and  $T^\otimes$ , respectively. Black, white, and star vertices correspond to row, column, and symbol labels of  $T^\diamond$ .*

*Proof.* Let  $(T^\diamond, T^\otimes)$  be a 3-homogeneous bitrade and let  $[\tau_1, \tau_2, \tau_3]$  be the  $\tau_i$  representation. By Condition (T1) and (T4) we see that  $\tau_1, \tau_2$ , and  $\tau_3$  satisfy the properties to be a combinatorial hypermap. Let  $[\sigma, \alpha, \varphi] = [\tau_1, \tau_2, \tau_3]$  and construct the associated hypermap using Construction 3.1. Apply Construction 3.3 so that the hypermap consists of shaded and unshaded triangles. Since  $z(\tau_1) = z(\tau_2) = z(\tau_3) = |T^\diamond|/3$  and  $|\Omega| = |T^\diamond|$  it follows that  $g = 1$  so the underlying surface is the torus. The fundamental group of the torus is  $\mathbb{Z} \times \mathbb{Z}$  so the covering surface is the Euclidean plane. By Construction 3.3, each shaded triangle is adjacent, edge-wise, to precisely one unshaded triangle, and vice-versa. The permutation  $\sigma$  acts on shaded triangles while  $\tau_1$  acts on elements of  $T^\diamond$  by Equation (1). We set  $\sigma = \tau_1$  so shaded triangles correspond to elements of  $T^\diamond$  and unshaded triangles correspond to elements of  $T^\otimes$ . Black vertices correspond to cycles of  $\tau_1$  which, in turn, correspond to row labels of  $T^\diamond$  (and similar for white and star vertices).  $\square$

**Example 3.6.** Figure 4 shows the tessellation for the 3-homogeneous bitrade of Example 2.6. Identifying opposite sides of the parallelogram marked by thick grey lines gives the torus. With regards to Theorem 1.1, we can partition  $T^\diamond$  into three transversals

$$\begin{aligned}\mathfrak{T}_1 &= \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4)\}, \\ \mathfrak{T}_2 &= \{(1, 4, 2), (2, 1, 3), (3, 2, 4), (4, 3, 1)\}, \\ \mathfrak{T}_3 &= \{(1, 2, 3), (2, 3, 4), (3, 4, 1), (4, 1, 2)\}\end{aligned}$$

where  $T^\diamond = \mathfrak{T}_1 \cup \mathfrak{T}_2 \cup \mathfrak{T}_3$ . These transversals may be located geometrically in Figure 4:  $\mathfrak{T}_1$  is made up of shaded triangles located directly above a  $\bullet$  vertex,  $\mathfrak{T}_2$  is made up of shaded triangles located directly to the lower-left of a  $\bullet$  vertex, and  $\mathfrak{T}_3$  is made up of shaded triangles located directly to the lower-right of a  $\bullet$  vertex.

## 4 The geometric proof

In this section we provide the geometric proof of Theorem 1.1. Let  $(T^\diamond, T^\otimes)$  be a 3-homogeneous bitrade. Apply Lemma 3.5 to obtain the labelled tessellation of the Euclidean plane for  $(T^\diamond, T^\otimes)$ . Without loss of generality, let the triangles have unit length sides. Let  $t$  be an unshaded triangle in the tessellation. Define three actions  $\rho_i$  on  $t$ :  $\rho_1$  rotates  $t$  by angle  $2\pi/3$  anticlockwise around its  $\bullet$  vertex;  $\rho_2$  rotates  $t$  by angle  $2\pi/3$  anticlockwise around its  $\circ$  vertex;  $\rho_3$  rotates  $t$  by angle  $2\pi/3$  anticlockwise around its  $\star$  vertex. The plane is tessellated by hexagons similar to those in Figure 4.

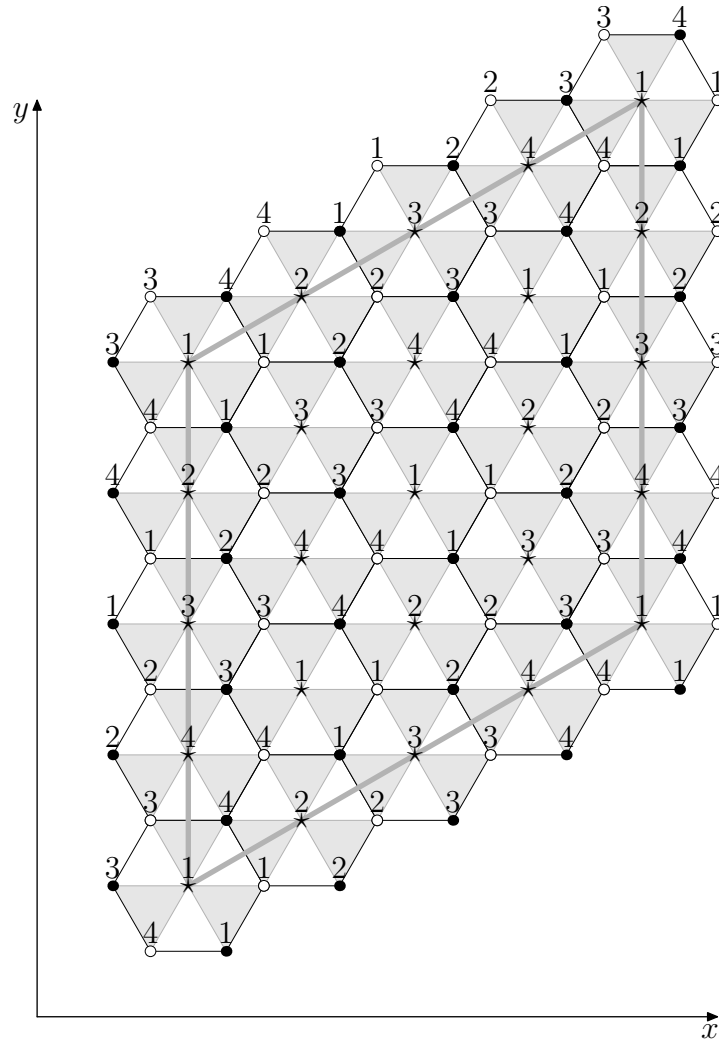


Figure 4: Canonical triangulation of the bitrade in Example 2.6. Identifying opposite sides of the solid grey parallelogram gives a torus.



Since  $(T^\circ, T^\otimes)$  is 3-homogeneous, it follows that there are three shaded triangles at each vertex, so  $\rho_i^3 = 1$  for  $1 \leq i \leq 3$  and  $\rho_1\rho_2\rho_3 = 1$ . These  $\rho_i$  induce a *triangle group*  $\Gamma$  which acts on the set of equilateral triangles  $\mathcal{T}$  of the tessellation:

$$\Gamma = \langle \rho_1, \rho_2, \rho_3 \mid \rho_1^3 = \rho_2^3 = \rho_3^3 = \rho_1\rho_2\rho_3 = 1 \rangle. \quad (4)$$

If  $[\tau_1, \tau_2, \tau_3]$  is the  $\tau_i$  representation for the bitrade in question, then we define the *cartographic group*  $G$  by:

$$G = \langle \tau_1, \tau_2, \tau_3 \mid \tau_1^3 = \tau_2^3 = \tau_3^3 = \tau_1\tau_2\tau_3 = \dots = 1 \rangle. \quad (5)$$

Note that  $\Gamma$  is an infinite group, acting on the tessellation of the Euclidean plane, while  $G$  is a finite permutation group, acting on the corresponding triangles on the identified surface (the torus). The group  $G$  has all of the defining relations for  $\Gamma$  so it is natural to define a group homomorphism  $\theta$  that sends  $\rho_i$  to  $\tau_i$  and the empty word  $1_\Gamma$  to the identity  $1_G$ . We then extend  $\theta$  to an arbitrary word by  $\rho_{i_1}\rho_{i_2}\dots\rho_{i_n} \mapsto \tau_{i_1}\tau_{i_2}\dots\tau_{i_n}$  where  $i_\ell \in \{1, 2, 3\}$  for  $1 \leq \ell \leq n$ .

To relate the group actions  $\mathcal{T} \times \Gamma \rightarrow \mathcal{T}$  and  $T^\circ \times G \rightarrow \Omega$  we form a map  $\psi: \mathcal{T} \rightarrow \Omega$ . Fix a shaded triangle  $t_0 \in \mathcal{T}$  and an entry  $x_0 \in \Omega$  and set  $t_0\psi = x_0$ . Then use  $\theta$  to extend  $\psi$  to any  $t \in \mathcal{T}$  by defining  $t\psi = x_0(\delta\theta)$  where  $t_0\delta = t$  for some  $\delta \in \Gamma$ .

**Lemma 4.1.** *The map  $\psi$  with base points  $t_0$  and  $x_0$  is well defined and commutes with the actions of  $\Gamma$  and  $G$ .*

*Proof.* Let  $\psi$  be defined as above for some fixed  $t_0, x_0$ . First we check that  $\psi$  is a well-defined map, namely that the choice of  $\delta$  for  $t_0\delta = t$  does not matter. Suppose that  $t_0\delta_1 = t = t_0\delta_2$ . Then  $t_0\delta_1\delta_2^{-1} = t_0$  so  $(\delta_1\delta_2^{-1})\theta = g \in G$  where  $x_0g = x_0$  (we can't assume that  $g$  is the identity in  $G$ , only that it fixes  $x_0$ ). Then

$$\begin{aligned} x_0g &= x_0(\delta_1\delta_2^{-1})\theta = x_0(\delta_1\theta)(\delta_2^{-1}\theta) = x_0 \\ \Rightarrow x_0(\delta_1\theta) &= x_0(\delta_2^{-1}\theta)^{-1} = x_0(\delta_2\theta) \\ \Rightarrow x_0(\delta_1\theta) &= x_0(\delta_2\theta) \end{aligned}$$

so  $t\psi$  takes the same value whether  $\delta_1$  or  $\delta_2$  was chosen. Hence  $\psi$  is well defined.

Next we check that  $\psi$  commutes with both group actions  $\nu$  and  $\eta$  of  $\Gamma$  and  $G$ , respectively. In other words, the following diagram must commute:

$$\begin{array}{ccc} \mathcal{T} \times \Gamma & \xrightarrow{\nu} & \mathcal{T} \\ \psi \times \theta \downarrow & & \downarrow \psi \\ \Omega \times G & \xrightarrow{\eta} & \Omega \end{array}$$

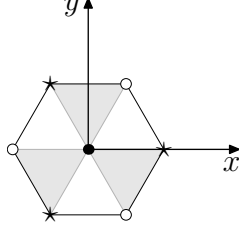


Figure 5: One possible placement of the  $x, y$  axes.

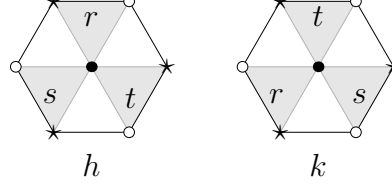


Figure 6: Inconsistently labelled hexagons due to  $r \in \mathfrak{T}_1 \cap \mathfrak{T}_2$ .

Choose  $t \in \mathcal{T}$ ,  $\xi \in \Gamma$ . Then

$$\begin{aligned} (t, \xi)\nu\psi &= (t\xi)\psi = x_0(\delta_1\theta) \\ (t, \xi)(\psi \times \theta)\eta &= (x_0(\delta_2\theta), \xi\theta)\eta = x_0(\delta_2\theta)(\xi\theta) = x_0((\delta_2\xi)\theta) \end{aligned}$$

where  $t_0\delta_1 = (t\xi)$  and  $t_0\delta_2 = t$ . Then  $t_0(\delta_2\xi) = t\xi = t_0\delta_1$  so  $\nu\psi = (\psi \times \theta)\eta$  and the diagram commutes.  $\square$

The tessellation lies on the Euclidean plane and we are free to place the  $x, y$  axes as we wish. We will choose one of three placements: that shown in Figure 5, or the rotation of those axes by angle  $2\pi/3$  or  $-2\pi/3$ . In other words, the origin is always at a  $\bullet$  vertex, the  $x$  axis always ‘points’ through a  $\star$  vertex, and the  $y$  axis (in the positive direction) bisects a shaded triangle.

**Definition 4.2.** Geometrically define three subsets  $\mathfrak{T}_i \subset T^\circ$  as follows:

$$\begin{aligned} \mathfrak{T}_1 &= \{t\psi \mid t \text{ is the shaded triangle immediately above } v \text{ with } v \text{ as its } \bullet \text{ vertex}\} \\ \mathfrak{T}_2 &= \{t\psi \mid t \text{ is the shaded triangle immediately to} \\ &\quad \text{the lower-left of } v \text{ with } v \text{ as its } \bullet \text{ vertex}\} \\ \mathfrak{T}_3 &= \{t\psi \mid t \text{ is the shaded triangle immediately to} \\ &\quad \text{the lower-right of } v \text{ with } v \text{ as its } \bullet \text{ vertex}\} \end{aligned}$$

where  $v$  ranges over all  $\bullet$  vertices.

In what follows it will be convenient to label a triangle  $t$  in the tessellation by  $t\psi$ . We now move on to showing that the  $\mathfrak{T}_i$  sets are mutually disjoint.

**Lemma 4.3.** *Let  $\mathfrak{T}_1$  and  $\mathfrak{T}_2$  be given as in Definition 4.2. Then  $\mathfrak{T}_1 \cap \mathfrak{T}_2 = \emptyset$ .*

*Proof.* Suppose, for contradiction, that there exists a triple  $r \in T^\circ$  such that  $r \in \mathfrak{T}_1 \cap \mathfrak{T}_2$ . Since  $\tau_1$  has no fixed-point (Condition (T3) in the definition of a bitrade) there must be a 3-cycle  $(r, s, t)$  in  $\tau_1$  for some  $s, t \in T^\circ$ . If  $u\psi = r$  for some shaded triangle  $u \in \mathcal{T}$  then it must be that  $(u\rho_1)\psi = r\tau_1 = s$  and  $(u\rho_1^2)\psi = r\tau_1^2 = t$ . Recalling that  $\rho_1$  is rotation about a  $\bullet$  vertex in the anticlockwise direction, the labelled tessellation must have hexagons like those

shown in Figure 6. The hexagon  $h$  has  $r \in \mathfrak{T}_1$  while the hexagon  $k$  has  $r \in \mathfrak{T}_2$ . The order of  $r, s, t$  is forced by  $\psi$ .

Let each side of a triangle in the tessellation have unit length. Without loss of generality we can place the  $x, y$  axes on the tessellation (as in Figure 5) so that the  $\bullet$  vertex of  $h$  is at  $(0, 0)$  and the  $\bullet$  vertex of  $k$  is at  $(x, y)$ . Then we have a Euclidean distance  $d(h, k) = \sqrt{x^2 + y^2}$  which we assume to be minimal. We then show that there exists another pair of inconsistently labelled hexagons  $h'$  and  $k'$  such that  $d(h', k') < d(h, k)$  except for a few cases in which contradictions arise with respect to the bitrade itself. In the limiting case we get  $d(h', k') = 0$  which implies that  $\tau_1$  has a fixed point  $r$ . There are four main cases to check, each with three subcases **a**, **b**, and **c**. Each of the **a** and **b** cases cover an infinite part of the plane so we use various constructions to find  $h', k'$  such that  $d(h', k') < d(h, k)$ . The **c** cases are finite and provide the required local contradictions.

**Case 1:**  $x, y \geq 0$ .

**Case 1a:**  $x > 3/2, y \geq 0$ . Suppose that  $t\tau_1\tau_2 = w$  and consider the action of  $\rho_1\rho_2$  on the triangles labelled  $t$  as shown in Figure 7. Recall that  $\rho_1$  is rotation to the next shaded triangle in an anticlockwise direction around a  $\bullet$  vertex, and  $\rho_2$  is rotation around a  $\circ$  vertex. We find  $d(h', k')$  and factor out the  $d(h, k)$  term:

$$\begin{aligned} d(h', k')^2 &= (x - 3)^2 + y^2 \\ &= x^2 - 6x + 9 + y^2 \\ &= d(h, k)^2 - 6x + 9. \end{aligned}$$

Now  $d(h', k')^2 - d(h, k)^2 = -6x + 9 < 0$  since  $x > 3/2$  so  $d(h', k') < d(h, k)$ . The last step is to observe that there must be a cycle  $(w, w', w'')$  in  $\tau_1$  as shown above. Then  $w'' \in \mathfrak{T}_1 \cap \mathfrak{T}_2$ , so  $h'$  and  $k'$  are a closer pair of inconsistent hexagons. This completes Case 1a.

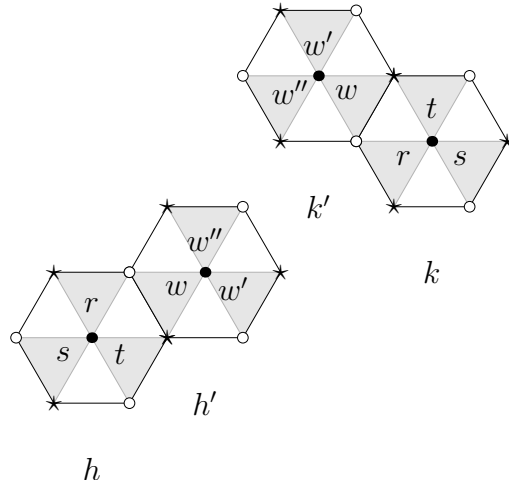


Figure 7: Case 1a.

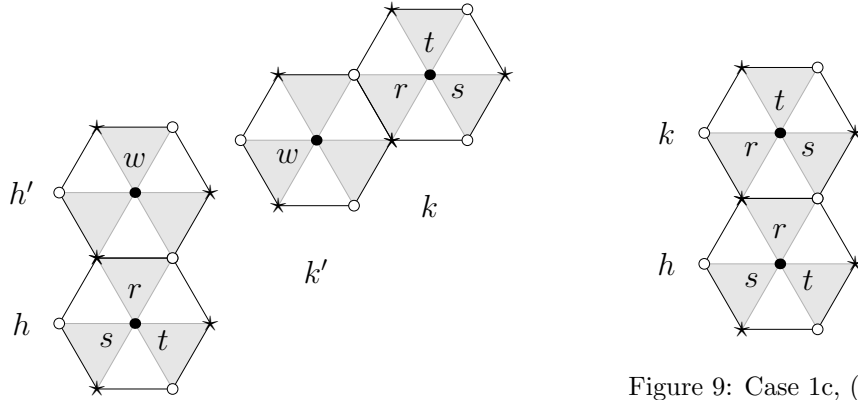


Figure 8: Case 1b.

Figure 9: Case 1c, (ii).

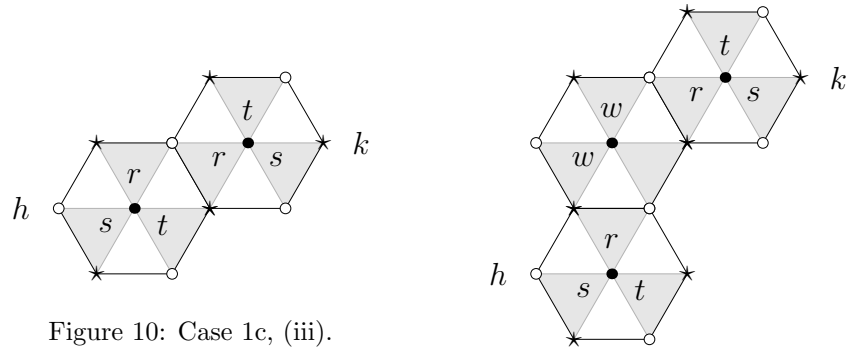


Figure 10: Case 1c, (iii).

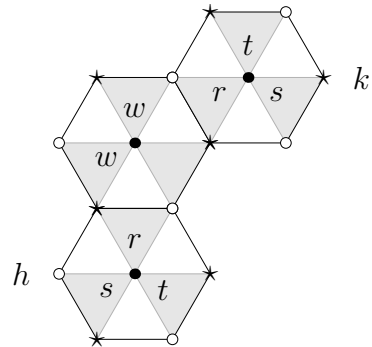


Figure 11: Case 1c, (iv).

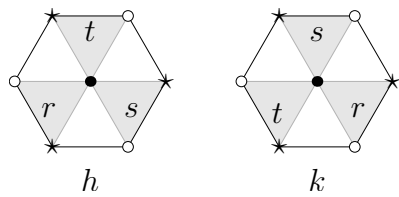


Figure 12: Inconsistently labelled hexagons due to  $r \in \mathfrak{T}_2 \cap \mathfrak{T}_3$ .

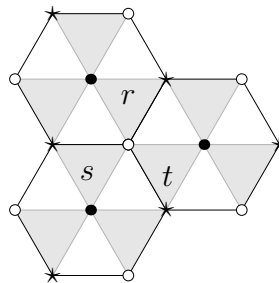


Figure 13: Consistency at a  $\circ$  vertex.

**Case 1b:**  $0 \leq x \leq 3/2$ ,  $y \geq 4\frac{\sqrt{3}}{2}$ . Suppose that  $r\tau_3\tau_1^{-1} = w$  and consider the action of  $\rho_3\rho_1^{-1}$  on the triangles labelled  $r$  as in Figure 8. We calculate the distance between  $h'$  and  $k'$ :

$$\begin{aligned} d(h', k')^2 &= (x - 3/2)^2 + \left(y - 3\frac{\sqrt{3}}{2}\right)^2 \\ &= x^2 - 3x + 9/4 + y^2 - 3\sqrt{3}y + 27/4 \\ &= d(h, k)^2 - 3x + 9 - 3\sqrt{3}y \end{aligned}$$

We need  $-3x + 9 - 3\sqrt{3}y < 0$  which simplifies to  $-x + 3 - \sqrt{3}y < 0$ . By assumption,  $-x + 3 - \sqrt{3}y \leq -x + 3 - 4\sqrt{3}\frac{\sqrt{3}}{2} \leq -x + 3 - 6 = -x - 3 < 0$  and this completes Case 1b.

**Case 1c:**  $0 \leq x \leq 3/2$ ,  $0 \leq y < 4\frac{\sqrt{3}}{2}$ . Here we deal with local cases which give rise to contradictions. Note that some  $(x, y) \in \{0, \frac{3}{2}\} \times \{0, 1\frac{\sqrt{3}}{2}, 2\frac{\sqrt{3}}{2}, 3\frac{\sqrt{3}}{2}\}$  do not correspond to valid hexagon positions, e.g. there is no hexagon centred at  $(0, 1\frac{\sqrt{3}}{2})$ . The valid cases are as follows:

- (i)  $x = 0, y = 0$ : In this case  $h$  and  $k$  are the same hexagon, implying that  $r\tau_1 = r$  so  $\tau_1$  has a fixed-point which contradicts (T3) in the definition of a bitrade
- (ii)  $x = 0, y = 2\frac{\sqrt{3}}{2}$ : In this case  $\tau_3$  has a fixed-point  $r$ , contradicting (T3) (see Figure 9).
- (iii)  $x = 3/2, y = \frac{\sqrt{3}}{2}$ : Here we find a fixed-point of  $\tau_2$ , contradicting (T3) (see Figure 10).
- (iv)  $x = 3/2, y = 3\frac{\sqrt{3}}{2}$ : If  $r\tau_3\tau_1^{-1} = w$  then the action of  $\rho_3\rho_1^{-1}$  on the triangles labelled  $r$  shows that  $\tau_1$  has a fixed-point as shown in Figure 11.

Cases 2, 3, and 4 are very similar. For each sub-case of type **a** and **b** we state the  $\rho_i\rho_j$  word along with the corresponding action as  $\tau_i\tau_j$ . Each sub-case of type **c** gives an immediate contradiction in the form of a fixed point for some  $\tau_1, \tau_2$ , or  $\tau_3$ .

**Case 2:**  $x, y \leq 0$ .

**Case 2a:**  $x < -3/2, y \leq 0$ . Use  $\rho_3\rho_1^{-1}$  where  $s\tau_3\tau_1^{-1} = w$ .

**Case 2b:**  $-3/2 \leq x \leq 0, y < -3\sqrt{3}/2$ . Use  $\rho_1\rho_2$  where  $s\tau_1\tau_2 = w$ .

**Case 2c:**  $-3/2 \leq x \leq 0, -3\sqrt{3}/2 \leq y \leq 0$ .

**Case 3:**  $x \geq 0, y \leq 0$ .

**Case 3a:**  $x > 3/2, y \leq 0$ . Use  $\rho_2\rho_1^{-1}$  where  $r\tau_2\tau_1^{-1} = w$ .

**Case 3b:**  $0 \leq x \leq 3/2, y < -3\sqrt{3}/2$ . Use  $\rho_3\rho_1^{-1}$  where  $t\tau_3\tau_1^{-1} = w$ .

**Case 3c:**  $0 \leq x \leq 3/2, -3\sqrt{3}/2 \leq y \leq 0$ .

**Case 4:**  $x \leq 0, y \geq 0$ .

**Case 4a:**  $x < -3/2, y \geq 0$ . Use  $\rho_2^{-1}\rho_1$ , where  $s\tau_2^{-1}\tau_1 = w$ .

**Case 4b:**  $-3/2 \leq x \leq 0, y > 2\sqrt{3}/2$ . Use  $\rho_2\rho_1^{-1}$ , where  $s\tau_2\tau_1^{-1} = w$ .  
**Case 4c:**  $-3/2 \leq x \leq 0, 0 \leq y \leq 2\sqrt{3}/2$ .

This completes the proof of Lemma 4.3  $\square$

**Corollary 4.4.** *Let  $\mathfrak{T}_1, \mathfrak{T}_2$ , and  $\mathfrak{T}_3$ , be given as in Definition 4.2. Then the  $\mathfrak{T}_i$  are mutually disjoint.*

*Proof.* Suppose that there exists  $r \in T^\diamond$  such that  $r \in \mathfrak{T}_2 \cap \mathfrak{T}_3$ . Since  $\tau_1$  is fixed-point-free it must contain a 3-cycle  $(r, s, t)$  for some  $s, t \in T^\diamond$ . Then the inconsistent hexagons are as shown in Figure 12. We see that  $t \in \mathfrak{T}_1 \cap \mathfrak{T}_2$ , so by Lemma 4.3 we have a contradiction. The case where  $r \in \mathfrak{T}_1 \cap \mathfrak{T}_3$  is similar.  $\square$

**Corollary 4.5.** *Let  $\mathfrak{T}_1, \mathfrak{T}_2$ , and  $\mathfrak{T}_3$ , be given as in Definition 4.2. Then the set  $\{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3\}$  is a partition of  $T^\diamond$ .*

*Proof.* By Corollary 4.4 the  $\mathfrak{T}_i$  sets are mutually disjoint. By assumption, the group  $G = \langle \tau_1, \tau_2, \tau_3 \rangle$  acts transitively on  $T^\diamond$ . The group homomorphism  $\theta$  given by  $\rho_i \mapsto \tau_i$  is actually a group epimorphism. With Lemma 4.1 it follows that each  $r \in T^\diamond$  will be an element of some  $\mathfrak{T}_i$  set. Hence  $\{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3\}$  is a partition of  $T^\diamond$ .  $\square$

**Lemma 4.6.** *The three sets  $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$  as defined in Definition 4.2 are transversals.*

*Proof.* In Lemma 4.3 and Corollary 4.5 we constructed the partition by dividing up elements of  $T^\diamond$  around a  $\bullet$  vertex, so it is impossible for any  $\mathfrak{T}_i$  to have more than one element from a row of  $T^\diamond$  (in particular this would imply a fixed point of  $\tau_1$ ). Conversely, suppose that a cycle  $(r, s, t)$  exists in  $\tau_2$  and is labelled as shown in Figure 13. Now  $r \in \mathfrak{T}_3, s \in \mathfrak{T}_1, t \in \mathfrak{T}_2$  according to the labelling induced around  $\bullet$  vertices. If another hexagon centred at a  $\circ$  vertex was inconsistently labelled then we would have an inconsistent labelling around  $\bullet$  vertices, contradicting Corollary 4.5. Similarly, labellings around  $\star$  vertices are consistent.  $\square$

Corollary 4.5 and Lemma 4.6 give the main result, Theorem 1.1. We note that, in general, the covering surface will be spherical, Euclidean, or hyperbolic. Most (large) bitrades will be hyperbolic, and we expect that future work will derive combinatorial properties of hyperbolic bitrades from their geometrical representation.

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